

Test 3

Section I, Part A

Allotted Time: 62 Minutes

Number of Questions: 29 Questions

Calculator Utilization: The use of a calculator is not permitted for this part of the test.

Instructions: Work through each of the problems, using the space provided to jot down calculations or notes. Carefully examine the answer choices for each question and determine which is the most accurate. Once you've selected the best answer, fill in the matching oval on the answer sheet. Keep in mind that anything written in this booklet won't be graded. Don't get stuck on any single question for too long. In this test, unless stated otherwise, assume that the domain of a function f includes all real values of x that result in $f(x)$ being a real number:

1. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} =$

- a. 0
- b. 3
- c. 4
- d. 6

2. The function g is differentiable, and some values of $g(x)$ are shown in the table below.

x	1.8	2.0	2.2
$g(x)$	3.61	4.00	4.41

Which of the following is the best estimate of $g'(2)$?

- a. 1.6
- b. 1.8
- c. 2.0
- d. 2.2

3. The derivative of a function f is

$$f'(x) = (x - 1)(x + 2)^2.$$

Which of the following is true about f ?

- a. f is increasing on $(-\infty, -2)$ and $(1, \infty)$.
 - b. f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.
 - c. f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.
 - d. f is increasing on $(-2, 1)$ and decreasing on $(1, \infty)$.
4. Let $y = (\sin(3x))^2$. Find $\frac{dy}{dx}$.
- a. $2 \sin(3x) \cos(3x)$
 - b. $3 \sin^2(3x)$
 - c. $6 \sin(3x) \cos(3x)$
 - d. $6 \sin^2(3x)$

5. Let

$$f(x) = \begin{cases} 2x + 1, & x < 1 \\ 3, & x = 1 \\ x^2, & x > 1 \end{cases}$$

Which of the following statements is true?

- a. $\lim_{x \rightarrow 1} f(x) = 3$ and f is continuous at $x = 1$.
- b. $\lim_{x \rightarrow 1^-} f(x) = 3$ and $\lim_{x \rightarrow 1^+} f(x) = 1$, so $\lim_{x \rightarrow 1} f(x)$ does not exist.
- c. $\lim_{x \rightarrow 1} f(x) = 1$ and f is continuous at $x = 1$.
- d. $\lim_{x \rightarrow 1^-} f(x) = 1$ and $\lim_{x \rightarrow 1^+} f(x) = 3$, so $\lim_{x \rightarrow 1} f(x) = 2$.

6. The position of a particle moving along a line is given by

$$s(t) = 2t^2 + 3t$$

in meters, where t is in seconds. What is the instantaneous velocity of the particle at $t = 4$?

- a. 14 m/s
- b. 16 m/s
- c. 18 m/s
- d. 19 m/s

7. Let

$$h(x) = \frac{x^2 + 1}{x - 3}.$$

Find $h'(x)$.

- a. $\frac{2x(x-3)+(x^2+1)}{(x-3)^2}$
- b. $\frac{2x(x-3)-(x^2+1)}{(x-3)^2}$
- c. $\frac{x^2-6x-1}{x-3}$
- d. $\frac{x^2-6x-1}{(x-3)^2}$

8. Suppose

$$f''(x) = (x - 2)(x + 1)$$

Which statement about the concavity of f is true?

- a. f is concave up on $(-1, 2)$ and concave down on $(-\infty, -1) \cup (2, \infty)$.
- b. f is concave up on $(-\infty, -1) \cup (2, \infty)$ and concave down on $(-1, 2)$.
- c. f is concave up on $(-\infty, 2)$ and concave down on $(2, \infty)$.
- d. f is concave down on $(-\infty, -1) \cup (2, \infty)$ and concave up on $(-1, 2)$.

9. Find

$$\int (6x^2 - 4 \cos x + 3) dx.$$

- a. $2x^3 - 4 \sin x + 3x + C$
- b. $2x^3 + 4 \sin x + 3x + C$
- c. $3x^3 - 4 \sin x + 3x + C$
- d. $3x^3 - 4 \cos x + 3x + C$

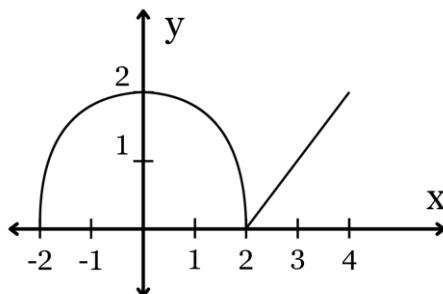
10. The equation of a circle is

$$x^2 + y^2 = 25.$$

At the point $(3, 4)$, what is $\frac{dy}{dx}$?

- a. $-\frac{4}{3}$
- b. $-\frac{3}{4}$
- c. $\frac{3}{4}$
- d. $\frac{4}{3}$

11. The graph of f is shown below (assume that f is a semi-circle for $-2 \leq x \leq 2$).



What is the area of the region bounded by the graph of f and the x -axis from $x = -2$ to $x = 4$?

- a. 2π
- b. $2\pi + 1$
- c. $2\pi + 2$
- d. $4\pi + 2$

12. Let

$$f(x) = \begin{cases} kx + 2, & x < 3 \\ x^2 - 1, & x \geq 3 \end{cases}$$

What value of k makes f continuous at $x = 3$?

- a. 0
- b. $\frac{2}{3}$
- c. 2
- d. $\frac{8}{3}$

13. The graph of f is composed of two line segments: from $(0, 0)$ to $(2, 4)$, and from $(2, 4)$ to $(4, 0)$.

What is

$$\int_0^4 f(x) \, dx ?$$

- a. 4
- b. 6
- c. 8
- d. 10

14. The graph of $f'(x)$ has these properties:

- $f'(x) > 0$ for $x < 1$
- $f'(1) = 0$
- $f'(x) < 0$ for $1 < x < 3$
- $f'(3) = 0$
- $f'(x) > 0$ for $x > 3$

Which statement is true?

- a. f has a local minimum at $x = 1$ and a local maximum at $x = 3$.
- b. f has a local maximum at $x = 1$ and a local minimum at $x = 3$.
- c. f has local minima at both $x = 1$ and $x = 3$.
- d. f has local maxima at both $x = 1$ and $x = 3$.

15. A slope field corresponds to the differential equation

$$\frac{dy}{dx} = x - y$$

At which point is the slope of the solution curve zero?

- a. $(0, 0)$
 - b. $(1, 0)$
 - c. $(0, 1)$
 - d. $(2, 2)$
16. A spherical balloon is being inflated so that its radius increases at a constant rate of 2 cm/s. At the instant when the radius is 5 cm, what is the rate of change of the volume?
- a. $20\pi \text{ cm}^3/\text{s}$
 - b. $40\pi \text{ cm}^3/\text{s}$
 - c. $100\pi \text{ cm}^3/\text{s}$
 - d. $200\pi \text{ cm}^3/\text{s}$
17. If $f(x) = 3x^4 - 2 \cos x$, what is $f'(x)$?
- a. $12x^3 + 2 \sin x$
 - b. $12x^3 - 2 \sin x$
 - c. $12x^4 + 2 \sin x$
 - d. $12x^3 - 2 \cos x$
18. The graph of the velocity of a particle $v(t)$ is piecewise linear:
- From $t = 0$ to $t = 2$, v decreases linearly from 4 to 0.
 - From $t = 2$ to $t = 4$, v decreases linearly from 0 to -2 .

What is the displacement of the particle on $[0, 4]$?

- a. -2
- b. 0
- c. 1
- d. 2

19. Which definite integral is represented by

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{2i}{n}\right)^2 \cdot \frac{2}{n} ?$$

- a. $\int_0^2 (1+x)^2 dx$
- b. $\int_0^2 (1+2x)^2 dx$
- c. $\int_1^3 x^2 dx$
- d. $\int_1^5 x^2 dx$

20. If $f(2) = 5$ and $f'(2) = 4$, what is $(f^{-1})'(5)$?

- a. $\frac{1}{2}$
- b. $\frac{1}{4}$
- c. 4
- d. $-\frac{1}{4}$

21. A rectangle has its base on the x -axis and its upper corners on the parabola $y = 12 - x^2$. If the rectangle is symmetric about the y -axis, what is the maximum possible area of the rectangle?

- a. 32
- b. 36
- c. 40
- d. 48

22. Find the area of the region bounded by $y = x$ and $y = x^2$.

- a. $\frac{1}{3}$
- b. $\frac{1}{6}$
- c. $\frac{1}{2}$
- d. $\frac{2}{3}$

23. A function y satisfies

$$\frac{dy}{dx} = 3y, \quad y(0) = 2.$$

What is $y(x)$?

- a. $y = 3e^{2x}$
- b. $y = 2e^{3x}$
- c. $y = e^{6x}$
- d. $y = 2e^x$

24. Let f be differentiable with $f(4) = 10$ and $f'(4) = -2$. Use linearization to approximate $f(4.1)$.

- a. 9.5
- b. 9.6
- c. 9.7
- d. 9.8

25. Let

$$F(x) = \int_1^{x^2} \sqrt{1+t^3} \, dt.$$

What is $F'(x)$?

- a. $\sqrt{1+x^3}$
- b. $2x\sqrt{1+x^3}$
- c. $\sqrt{1+x^6}$
- d. $2x\sqrt{1+x^6}$

26. The region bounded by $y = x^2$, $y = 0$, and $x = 2$ is revolved about the x -axis. What is the volume of the resulting solid?

- a. $\frac{32\pi}{5}$
- b. $\frac{16\pi}{5}$
- c. $\frac{64\pi}{5}$
- d. $\frac{32\pi}{3}$

27. Let $f(x) = x^2 - 4x + 1$ on $[0, 4]$. Which value of c satisfies the conclusion of the Mean Value Theorem?

- a. 1
- b. 2
- c. 3
- d. No such c exists

28. Evaluate

$$\int_0^1 2x(1 + x^2)^3 dx.$$

- a. $\frac{15}{2}$
- b. 4
- c. $\frac{15}{4}$
- d. $\frac{1}{4}$

29. Find the average value of $f(x) = x^2$ on $[0, 2]$.

- a. $\frac{2}{3}$
- b. $\frac{8}{3}$
- c. $\frac{4}{3}$
- d. $\frac{16}{3}$

Section I, Part B

Allotted Time: 38 Minutes

Number of Questions: 13 Questions

Calculator Utilization: The use of a calculator is permitted for the questions in this part of the test.

Instructions: For each problem below, solve using the space provided for scratch work. Review the format of the answer options, select the best choice, and mark the corresponding oval on your answer sheet. Note that any work written in the test booklet will not receive credit, so focus on completing the answer sheet. Avoid spending excessive time on any single question. The exact numerical answer may not always be included in the answer choices. In such cases, select the option that most closely approximates the exact value. Unless stated otherwise, the domain of a function is assumed to include all real numbers x for which $f(x)$ produces a real number.

30. The function f is defined for all $x \neq 2$. Values of $f(x)$ near $x = 2$ are shown:

x	1.9	1.99	2.01	2.1	2.001
$f(x)$	4.21	4.0201	4.0201	4.21	4.002001

Based on the table, what is the best estimate of $\lim_{x \rightarrow 2} f(x)$?

- a. 4.00
 - b. 4.02
 - c. 4.21
 - d. 4.04
31. Let $f(x) = \sqrt{x} \cdot \sin(x)$. Use a calculator to approximate $f'(4)$.
- a. -1.50
 - b. -0.06
 - c. 1.35
 - d. 2.31

32. The temperature of an object is $T(t)$ (in $^{\circ}\text{C}$), and the pressure inside the object is $P(T) = T^2 + 1$.

At time $t = 5$, $T(5) = 3$ and $T'(5) = -0.4$. What is $\frac{dP}{dt}$ at $t = 5$?

- a. -0.8
- b. 0.8
- c. -2.4
- d. 2.4

33. A particle's position $s(t)$ (meters) at time t (seconds) is shown:

t	1.8	1.9	2.0	2.1	2.2
$s(t)$	5.40	5.85	6.34	6.87	7.44

Which is the best estimate of the instantaneous velocity $s'(2)$?

- a. 4.8
- b. 5.1
- c. 5.5
- d. 5.7

34. Let $f(x) = x^4 - 6x^3 + 9x^2 + 1$ on $[0, 4]$. Use a calculator to determine the absolute minimum value of f on the interval.

- a. -2
- b. 0
- c. 2
- d. 1

35. Values of a continuous function g are given:

x	0	1	2	3	4
$g(x)$	2	3	5	6	8

Using the trapezoidal rule with 4 subintervals, approximate $\int_0^4 g(x) \, dx$.

- a. 18
- b. 19
- c. 20
- d. 21

36. A function y satisfies

$$\frac{dy}{dx} = \frac{x}{1+y^2}, \quad y(0) = 1.$$

Use the differential equation and initial condition to find $y(2)$. (Round to the nearest hundredth.)

- a. 1.45
- b. 1.58
- c. 1.70
- d. 1.85

37. The region enclosed by $y = \ln(x)$, $y = 0$, $x = 1$, and $x = 4$ is revolved about the x -axis. Which expression gives the volume, and what is its approximate value?

- a. $\pi \int_1^4 \ln(x) \, dx \approx 8.2$
- b. $\pi \int_1^4 [\ln(x)]^2 \, dx \approx 8.0$
- c. $\pi \int_1^4 [\ln(x)]^2 \, dx \approx 8.2$
- d. $\pi \int_1^4 \ln(x) \, dx \approx 8.0$

38. Let $f(x) = \frac{1}{x^2-4}$. Which statement is true?

- a. $f(x)$ has vertical asymptotes at $x = \pm 2$, and $\lim_{x \rightarrow 2^-} f(x) = +\infty$.
- b. $f(x)$ has vertical asymptotes at $x = \pm 2$, and $\lim_{x \rightarrow 2^-} f(x) = -\infty$.
- c. $f(x)$ has a vertical asymptote only at $x = 2$.
- d. $f(x)$ has no vertical asymptotes.

39. Let $f(x) = x \cos(x)$. Use a calculator to approximate the value of x in $[0, 3]$ where $f'(x) = 0$.

- a. 0.86
- b. 1.57
- c. 2.03
- d. 2.46

40. A curve is defined implicitly by

$$x^2 + xy + y^2 = 7.$$

At the point $(1.2, 2.0)$, approximate $\frac{dy}{dx}$.

- a. -0.58
- b. -0.85
- c. 0.58
- d. 0.85

41. A company's profit (in thousands of dollars) is modeled by

$$P(t) = -t^3 + 6t^2 + 15t$$

for $0 \leq t \leq 6$, where t is time in months. At what time t is the profit maximized?

- a. $t = 2$
- b. $t = 3$
- c. $t = 4$
- d. $t = 5$

42. A function f is twice differentiable. A graphing calculator shows that $f''(x)$ crosses the x -axis at $x \approx 1.1$, $x \approx 2.7$, and $x \approx 4.2$, and $f''(x) > 0$ for $x < 1.1$ and $2.7 < x < 4.2$. How many inflection points does f have?
- a. 0
 - b. 1
 - c. 2
 - d. 3

Section II, Part A

Allotted Time: 30 Minutes

Number of Questions: 2 Questions

Calculator Utilization: The use of a calculator is permitted for the questions in this part of the test.

Instructions: Providing your set up for these problems is essential, as partial credit may be granted. If you include decimal approximations, ensure they are accurate to three decimal places.

1. A tank contains 50 gallons of water at time $t = 0$. Water flows into the tank at a rate of $R(t)$ gallons per minute for $0 \leq t \leq 10$. Selected values of $R(t)$ are shown in the table below.

t (min)	0	2	4	6	8	10
$R(t)$ (gal/min)	3.0	3.8	4.1	3.6	2.9	2.5

- a. Use the trapezoidal rule with the data from the table to approximate

$$\int_0^{10} R(t) dt$$

- b. Interpret the meaning of $\int_0^{10} R(t) dt$ in the context of the problem.
 c. Find the average value of $R(t)$ on the interval $[0, 10]$. Indicate units.
 d. Use your answer from part (a) to approximate the amount of water in the tank at $t = 10$.
2. A company's profit, in thousands of dollars, is modeled by the function

$$P(x) = -0.2x^3 + 3.5x^2 + 10 \ln(x + 1) - 8e^{0.15x}$$

for $0 \leq x \leq 15$, where x is the number of thousands of items produced and sold.

- a. Find $P'(x)$.
 b. Use a calculator to find all critical values of P on the interval $[0, 15]$. Give your answer(s) to three decimal places.
 c. Determine the value of x on $[0, 15]$ at which $P(x)$ is maximized. Justify your answer.
 d. Use your answer from part (c) to approximate the maximum profit (in dollars). Round to the nearest dollar.

Section II, Part B

Allotted Time: 60 Minutes

Number of Questions: 4 Questions

Calculator Utilization: The use of a calculator is not permitted for this part of the test.

3. Let f be the piecewise function

$$f(x) = \begin{cases} ax^2 + bx, & x \leq 1 \\ \ln(x) + c, & x > 1 \end{cases}$$

where a , b , and c are all constants.

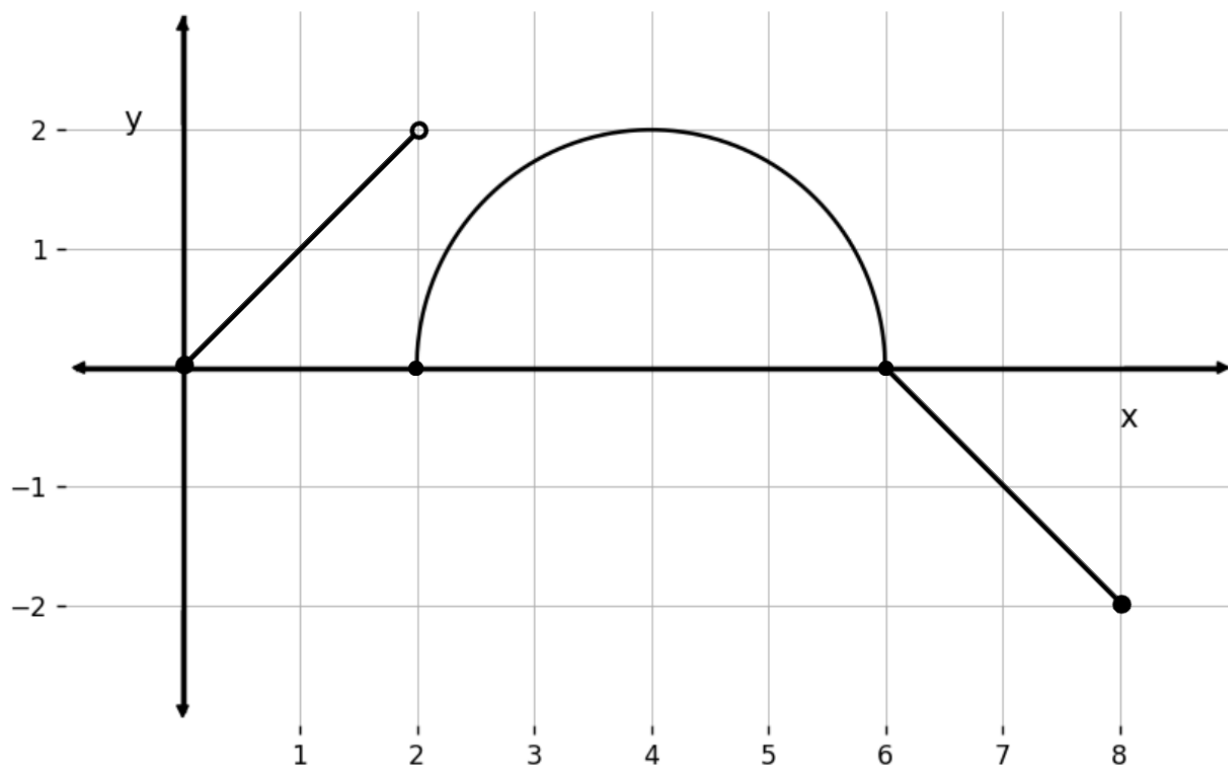
- Find the value of c in terms of a and b so that f is continuous at $x = 1$.
- Find a relationship between a and b so that f is differentiable at $x = 1$.
- Assuming f is continuous and differentiable at $x = 1$, write the equation of the tangent line to f at $x = 1$.
- For the values of a , b , and c that make f differentiable at $x = 1$, determine whether f is increasing on the interval $(0, 1)$. Justify your answer.

4. Let

$$f(x) = x^3 - 3x^2 - 9x + 5.$$

- Find $f'(x)$ and determine all critical values of f .
- On which intervals is f increasing? On which intervals is f decreasing? Justify your answer.
- Find all x -values where f has a point of inflection. Justify your answer.
- Verify that the hypotheses of the Mean Value Theorem are satisfied for f on the interval $[0, 3]$. Then find all values of c in $(0, 3)$ that satisfy the conclusion of the Mean Value Theorem.

5. Let f be a continuous function whose graph on the interval $[0, 8]$ is shown below:



Define

$$g(x) = \int_0^x f(t) \, dt.$$

- Find $g(2)$.
- Find $g'(7)$.
- On which interval(s) in $[0, 8]$ is g increasing? Justify your answer.
- Find the absolute maximum value of $g(x)$ on $[0, 8]$. Justify your answer.

6. A function y satisfies the differential equation

$$\frac{dy}{dx} = (x - 1)(y + 2).$$

- Show that $y = -2$ is a solution to the differential equation.
- Find the general solution to the differential equation.
- Find the particular solution that satisfies $y(0) = 0$.
- For the particular solution from part (c), determine the interval(s) of x for which the solution is increasing. Justify your answer.

Solutions

Section I, Part A

1. D	2. C	3. C	4. C	5. B
6. D	7. D	8. B	9. A	10. B
11. C	12. C	13. C	14. B	15. D
16. D	17. A	18. D	19. A	20. B
21. A	22. B	23. B	24. D	25. D
26. A	27. B	28. C	29. C	

1. Simply factor and simplify the expression.

$$\frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3 \quad (x \neq 3)$$

So

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = 3 + 3 = 6$$

2. The best way to estimate $g'(2)$ is to use the difference quotient around $x = 2$.

$$g'(2) \approx \frac{g(2.2) - g(1.8)}{2.2 - 1.8} = \frac{4.41 - 3.61}{0.4} = \frac{0.8}{0.4} = 2.0$$

3. The function f has critical points at $x = -2$ and $x = 1$ because those are the x values for which $f'(x) = 0$. Then we can create a gradient diagram using $f'(x)$.

Quantity of x	-3	-2	0	1	2
Direction of $f'(x)$	-	0	-	0	+
Gradient of graph $y = f(x)$					

So f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.

4. We can find $\frac{dy}{dx}$ by using the Chain Rule with $\sin(3x)$ being the inside function, and x^2 being the outside function. Finding the derivative of $\sin(3x)$ also requires the use of the Chain Rule with $3x$ being the inside function, and $\sin(x)$ being the outside function.

$$y = (\sin(3x))^2 \Rightarrow \frac{dy}{dx} = 2 \sin(3x) \cdot 3 \cdot \cos(3x) = 6 \sin(3x) \cos(3x)$$

5. Left-hand limit:

$$\lim_{x \rightarrow 1^-} f(x) = 2(1) + 1 = 3$$

Right-hand limit:

$$\lim_{x \rightarrow 1^+} f(x) = 1^2 = 1$$

Since the one-sided limits are not equal, $\lim_{x \rightarrow 1} f(x)$ does not exist. $f(1) = 3$, but the jump means the function is not continuous at $x = 1$.

6. Velocity is the rate of change, or the derivative, of position.

$$v(t) = s'(t) = 4t + 3$$

To find the instantaneous velocity of the particle when $t = 4$, simply evaluate $v(t)$ for $t = 4$.

$$v(4) = s'(4) = 4(4) + 3 = 16 + 3 = 19 \text{ m/s}$$

7. Using Quotient Rule:

$$h'(x) = \frac{(x-3) \cdot 2x - (x^2+1) \cdot 1}{(x-3)^2} = \frac{2x^2 - 6x - x^2 - 1}{(x-3)^2} = \frac{x^2 - 6x - 1}{(x-3)^2}$$

8. Set $f''(x) = (x-2)(x+1) = 0$ to find possible sign changes at $x = -1$ and $x = 2$. Testing one value in each interval shows whether the f is concave up or down throughout that interval.

- $x < -1$, e.g. -2 : $f''(-2) = (-4)(-1) > 0 \rightarrow$ concave up.
- $-1 < x < 2$, e.g. 0 : $f''(0) = (-2)(1) < 0 \rightarrow$ concave down.
- $x > 2$, e.g. 3 : $f''(3) = (1)(4) > 0 \rightarrow$ concave up.

So, f is concave up on $(-\infty, -1) \cup (2, \infty)$, concave down on $(-1, 2)$.

9.

$$\int 6x^2 \, dx = 2x^3, \quad \int -4 \cos x \, dx = -4 \sin x, \quad \int 3 \, dx = 3x.$$

So,

$$\int (6x^2 - 4 \cos x + 3) \, dx = 2x^3 - 4 \sin x + 3x + C.$$

10. Differentiate:

$$2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

At (3, 4):

$$\frac{dy}{dx} = -\frac{3}{4}.$$

11. Area = (area of semicircle of radius 2) + (area under line segment).

$$\text{Semicircle area} = \frac{1}{2} \pi (2^2) = 2\pi.$$

The line segment forms a right triangle with base 2 and height 2, area = $\frac{1}{2}(2)(2) = 2$.

$$\text{Total} = 2\pi + 2.$$

12. To make sure that f is continuous at $x = 3$, we need:

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$$

So,

$$3k + 2 = 3^2 - 1$$

$$3k + 2 = 8$$

$$3k = 6$$

$$k = 2$$

13. The region under the graph is one triangle with base 4 and height 4. Area = $\frac{1}{2}(4)(4) = 8$, so the integral is 8.14. A sign change in f' from positive to negative gives a local maximum (at $x = 1$); a sign change in f' from negative to positive gives a local minimum (at $x = 3$).15. Slope is 0 when $x - y = 0 \Rightarrow y = x$. Only (2, 2) satisfies that.

$$16. V = \frac{4}{3} \pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}. \text{ At } r = 5, \frac{dr}{dt} = 2: \frac{dV}{dt} = 4\pi(25)(2) = 200\pi.$$

$$17. \frac{d}{dx}(3x^4) = 12x^3 \text{ and } \frac{d}{dx}(-2 \cos x) = 2 \sin x.$$

18. Displacement is signed area under v .

- From $t = 0$ to $t = 2$: triangle area $= \frac{1}{2}(2)(4) = 4$.
- From $t = 2$ to $t = 4$: triangle below axis area $= -\frac{1}{2}(2)(2) = -2$.

Total: $4 + (-2) = 2$.

19. $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$ over $[0, 2]$. The term is $(1+x_i)^2$, so it matches $\int_0^2 (1+x)^2 dx$.

20. $(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(2)} = \frac{1}{4}$.

21. Let the upper right corner be $(x, 12 - x^2)$. Then width $= 2x$, height $= 12 - x^2$, so

$$A(x) = 2x(12 - x^2) = 24x - 2x^3, \quad A'(x) = 24 - 6x^2.$$

Set $A'(x) = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$. Then $A(2) = 4 \cdot 8 = 32$.

22. We need to find the intersections by setting $x = x^2 \Rightarrow x = 0, 1$. On $[0, 1]$, $x \geq x^2$.

$$\text{Area} = \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

23. First separate the variables, then integrate both sides of the equation and solve for y .

$$\frac{1}{y} dy = 3 dx \Rightarrow \ln|y| = 3x + C \Rightarrow y = Ce^{3x}.$$

Then use the given initial condition to solve for C .

$$y(0) = 2 \Rightarrow C = 2.$$

24. $f(4.1) \approx f(4) + f'(4)(0.1) = 10 + (-2)(0.1) = 9.8$.

25. According to Part 1 of the Fundamental Theorem of Calculus:

$$F'(x) = \sqrt{1 + (x^2)^3} \cdot \frac{d}{dx}(x^2) = \sqrt{1 + x^6} \cdot 2x.$$

26. This volume can be found using the disk method, with the radius of the disks being x^2 .

$$V = \pi \int_0^2 (x^2)^2 dx = \pi \int_0^2 x^4 dx = \pi \left[\frac{x^5}{5} \right]_0^2 = \pi \left(\frac{32}{5} \right) = \frac{32\pi}{5}.$$

27. First, find the average slope of f on $[0, 4]$.

$$\frac{f(4) - f(0)}{4 - 0} = \frac{(1) - (1)}{4} = 0.$$

So, we need to find the c that makes $f'(c) = 0$.

$$\begin{aligned} f'(x) &= 2x - 4 \\ f'(c) &= 2c - 4 = 0 \\ c &= 2 \end{aligned}$$

28. This can be evaluated using u -substitution. Let $u = 1 + x^2$, which means $du = 2x \, dx$. Then the bounds of the integral become: $x = 0 \rightarrow u = 1$, $x = 1 \rightarrow u = 2$.

$$\int_0^1 2x(1 + x^2)^3 \, dx = \int_1^2 u^3 \, du = \left[\frac{u^4}{4} \right]_1^2 = \frac{16 - 1}{4} = \frac{15}{4}$$

29. This is simply an application of the average value formula.

$$f_{\text{avg}} = \frac{1}{2 - 0} \int_0^2 x^2 \, dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2 = \frac{1}{2} \cdot \frac{8}{3} = \frac{4}{3}$$

Section I, Part B

30. B	31. A	32. C	33. B	34. D
35. B	36. C	37. C	38. B	39. A
40. B	41. D	42. D		

30. As x approaches 2 (e.g., 1.99 and 2.01), $f(x) \approx 4.0201$, so the limit is about 4.02.

31. We can find $f'(x)$ using the product rule and evaluate at $x = 4$. Alternatively, we can use the difference quotient with a small h to estimate:

$$f'(4) \approx \frac{f(4+h) - f(4)}{h}$$

(e.g., $h = 0.001$). This gives approximately -1.50 .

32. By chain rule:

$$\frac{dP}{dt} = \frac{dP}{dT} \cdot \frac{dT}{dt} = 2T \cdot T'$$

At $t = 5$:

$$2T \cdot T' = 2(3)(-0.4) = -2.4$$

33. To estimate the instantaneous velocity at $x = 2$, we can find the average velocity around $x = 2$.

$$s'(2) \approx \frac{s(2.1) - s(1.9)}{2.1 - 1.9} = \frac{6.87 - 5.85}{0.2} = 5.1$$

34. Graph f and use your calculator's minimum feature on $[0, 4]$. The lowest value is 1 (attained at $x = 0$ and $x = 3$).

35. Simply apply the trapezoidal rule formula with $\Delta x = 1$:

$$T = \frac{1}{2} [g(0) + 2(g(1) + g(2) + g(3)) + g(4)] = \frac{1}{2} [2 + 2(3 + 5 + 6) + 8] = 19.$$

36. Separate and integrate:

$$(1 + y^2) dy = x dx \Rightarrow y + \frac{y^3}{3} = \frac{x^2}{2} + C.$$

Use $y(0) = 1$:

$$1 + \frac{1}{3} = C = \frac{4}{3}.$$

At $x = 2$:

$$y + \frac{y^3}{3} = \frac{4}{2} + \frac{4}{3} = \frac{10}{3}$$

$$y^3 + 3y - 10 = 0$$

Solve this equation using your calculator to get: $y \approx 1.6989$, so $y(2) \approx 1.70$.

37. Washers about the x -axis have radius $\ln(x)$, so $V = \pi \int_1^4 [\ln(x)]^2 dx$. Evaluate this using your calculator to obtain $V \approx 8.2$.
38. $x^2 - 4 = (x - 2)(x + 2)$, so vertical asymptotes exist at $x = \pm 2$. As $x \rightarrow 2^-$, the denominator is negative and approaches 0, so $f(x) \rightarrow -\infty$.
39. Using product rule, we can see that $f'(x) = \cos(x) - x \sin(x)$. Then by graphing, we can use the calculator to solve $f'(x) = 0$ on $[0, 3]$. The solution is about 0.86.
40. We can solve this problem by differentiating implicitly and solving for y' , or $\frac{dy}{dx}$.

$$2x + (x \cdot y' + y) + 2y \cdot y' = 0$$

$$(x + 2y) y' = -2x - y$$

$$y' = \frac{-2x - y}{x + 2y}$$

Plugging in $(1.2, 2.0)$ gives $y' = -\frac{4.4}{5.2} \approx -0.85$.

41. Use a calculator to graph $P(t)$ and find the maximum of the function on $[0, 6]$ (or solve $P'(t) = 0$ and check the endpoints, $P(0)$ and $P(6)$). The maximum occurs at $t = 5$.
42. Inflection points of f occur where f'' changes sign. It changes sign at all three of its zeros, so f has 3 inflection points.

Section II, Part A

1. (a) The time step is $\Delta t = 2$. Using 5 trapezoids:

$$\begin{aligned}\int_0^{10} R(t) dt &\approx \frac{\Delta t}{2} [R(0) + 2(R(2) + R(4) + R(6) + R(8)) + R(10)] \\ &= \frac{2}{2} [3.0 + 2(3.8 + 4.1 + 3.6 + 2.9) + 2.5] \\ &= 3.0 + 28.8 + 2.5 = 34.3. \\ \int_0^{10} R(t) dt &\approx 34.3 \text{ gallons}\end{aligned}$$

- (b) $\int_0^{10} R(t) dt$ represents the total amount of water, in gallons, that flows into the tank over the first 10 minutes.

- (c) Average value:

$$R_{ave} = \frac{1}{10 - 0} \int_0^{10} R(t) dt.$$

Using part (a):

$$\begin{aligned}R_{ave} &\approx \frac{1}{10} (34.3) = 3.43. \\ R_{ave} &\approx 3.43 \text{ gal/min}\end{aligned}$$

- (d) Initial amount of water is 50 gallons. Approximate inflow after 10 minutes is 34.3 gallons. So, water at time $t = 10$ is roughly $50 + 34.3 = 84.3$ gallons.

2. (a) Differentiate:

$$\begin{aligned}P'(x) &= -0.6x^2 + 7x + \frac{10}{x+1} - 8(0.15)e^{0.15x} \\ P'(x) &= -0.6x^2 + 7x + \frac{10}{x+1} - 1.2e^{0.15x}\end{aligned}$$

- (b) Critical values occur where $P'(x) = 0$ (and where P' is undefined, but here $x \geq 0$ so $x + 1 \neq 0$).

Using a calculator to solve $P'(x) = 0$ on $[0, 15]$, the critical value is:

$$x \approx 10.857$$

- (c) Check the profit at the endpoints and the critical value using a calculator:

- $P(0) = 8e^0 = -8$
- $P(10.857) \approx 140.565$
- $P(15) \approx 64.324$

Since we compared the value of the function at the endpoints of the closed interval and all critical values on the interval, the greatest function value would be the maximum on the interval. So, profit is maximized at $x \approx 10.857$.

(d) Since P is in thousands of dollars, the maximum profit is

$$P(10.857) \approx 104.565 \text{ thousand dollars}$$

So, maximum profit is 104,565 dollars.

Section II, Part B

3. (a) Continuity requires $\lim_{x \rightarrow 1^-} f(x) = f(1) = \lim_{x \rightarrow 1^+} f(x)$.

Left-hand limit and value at $x = 1$:

$$\lim_{x \rightarrow 1^-} f(x) = f(1) = a(1)^2 + b(1) = a + b.$$

Right-hand limit:

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \ln(x) + c = \ln(1) + c = c.$$

Set equal,

$$\begin{aligned} a + b &= c \\ c &= a + b \end{aligned}$$

- (b) Differentiability requires matching derivatives from the left and right at $x = 1$.

Left derivative ($x \leq 1$):

$$\begin{aligned} \frac{d}{dx}(ax^2 + bx) &= 2ax + b \\ f'_-(1) &= 2a(1) + b = 2a + b \end{aligned}$$

Right derivative ($x > 1$):

$$\begin{aligned} \frac{d}{dx}(\ln(x) + c) &= \frac{1}{x} \\ f'_+(1) &= \frac{1}{1} = 1 \end{aligned}$$

Set equal:

$$2a + b = 1.$$

- (c) If differentiable, then slope is $f'(1) = 1$ (from part (b)), and the point is $(1, f(1))$ where $f(1) = a + b = c$ (from part (a)).

Using point-slope form:

$$\begin{aligned} y - (a + b) &= 1(x - 1) \quad \text{or} \quad y - c = 1(x - 1) \\ y &= x - 1 + a + b \quad \text{or} \quad y = x - 1 + c \end{aligned}$$

- (d) On $(0, 1)$, $f(x) = ax^2 + bx$, so $f'(x) = 2ax + b$. Using part (b), we know $b = 1 - 2a$.

Therefore:

$$f'(x) = 2ax + 1 - 2a = 1 + 2a(x - 1),$$

which is linear, so its minimum on $(0, 1)$ occurs at an endpoint of the interval. $f'(1) = 1$, which is positive, and $f'(0) = 1 - 2a$. Therefore, $f'(x) > 0$ for all $0 < x < 1$ exactly when $1 - 2a \geq 0$. In other words:

$$f \text{ is increasing on } (0, 1) \text{ if and only if } a \leq \frac{1}{2}.$$

4. (a) First, find the derivative:

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1).$$

Critical values occur where $f'(x) = 0$: $x = -1$ and $x = 3$.

$$f'(x) = 3(x - 3)(x + 1), \text{ critical values: } x = -1, 3$$

- (b) Use the sign of $f'(x) = 3(x - 3)(x + 1)$. This can be shown with a gradient diagram.

x	-2	-1	0	3	4
Sign of $f'(x)$	$+$	0	$-$	0	$+$
Direction of $f(x)$					

f is increasing on $(-\infty, -1) \cup (3, \infty)$, and decreasing on $(-1, 3)$.

- (c) First, find the second derivative:

$$f''(x) = 6x - 6 = 6(x - 1).$$

Inflection points may occur where $f''(x) = 0$: $x = 1$. Since $f''(x)$ changes sign from negative (when $x < 1$) to positive (when $x > 1$), f has an inflection point at $x = 1$.

- (d) Since f is a polynomial, it is continuous on $[0, 3]$ and differentiable on $(0, 3)$, so MVT applies.

Compute the average rate of change:

$$\frac{f(3) - f(0)}{3 - 0} = \frac{[27 - 27 - 27 + 5] - [5]}{3 - 0} = \frac{-27}{3} = -9.$$

Set $f'(c) = -9$:

$$f'(c) = 3c^2 - 6c - 9 = -9$$

$$3c^2 - 6c = 0$$

$$3c(c - 2) = 0$$

So, $c = 0$ or $c = 2$. Only $c = 2$ is in the interval $(0, 3)$. So,

$$c = 2.$$

5. (a) Compute $g(2)$:

$$g(2) = \int_0^2 f(t) \, dt.$$

From $x = 0$ to $x = 2$, f forms a right triangle with base 2 and height 2. Area:

$$\frac{1}{2}(2)(2) = 2.$$

$$g(2) = 2$$

- (b) By the Fundamental Theorem of Calculus,

$$g'(x) = f(x).$$

On $[6, 8]$, f is the line from $(6, 0)$ to $(8, -2)$ with slope $\frac{-2-0}{8-6} = -1$. So at $x = 7$, the value is $f(7) = -1$.

$$g'(7) = -1$$

- (c) g is increasing where $g'(x) \geq 0$, i.e., where $f(x) \geq 0$. From the graph, $f(x) \geq 0$ on $(0, 6)$.

On $(6, 8)$, $f(x) < 0$.

g is increasing on $(0, 6)$

- (d) An absolute maximum of g on a closed interval occurs at endpoints or where $g'(x) = 0$.

Since $g'(x) = f(x)$, candidates are $x = 0$, $x = 6$, and $x = 8$.

Compute $g(x)$ at these points:

- $g(0) = 0$

- $g(6) = \int_0^6 f(t) \, dt = (\text{area from } x = 0 \text{ to } 2) + (\text{area from } x = 2 \text{ to } 6).$

Area from $x = 0$ to 2 is a triangle with area 2.

Area from $x = 2$ to 6 is a semicircle with radius 2. Area $= \frac{1}{2}\pi(2)^2 = 2\pi$.

So,

$$g(6) = 2 + 2\pi$$

- $g(8) = g(6) + \int_6^8 f(t) \, dt$. From $x = 6$ to 8, f is below the axis forming a triangle with base 2 and height 2, so the signed area is -2 .

$$g(8) = (2 + 2\pi) + (-2) = 2\pi$$

Compare $g(0) = 0$, $g(6) = 2 + 2\pi$, and $g(8) = 2\pi$.

The absolute maximum value of g on $[0, 8]$ is $2 + 2\pi$ (at $x = 6$).

6. (a) If $y = -2$, we can take the derivative of both sides with respect to x so find $\frac{dy}{dx} = 0$. Plug these values into the differential equation to verify the solution.

$$\begin{aligned}\frac{dy}{dx} &= (x-1)(y+2) \\ 0 &= (x-1)((-2)+2) \\ 0 &= (x-1)(0) \\ 0 &= 0\end{aligned}$$

- (b) Separate the variables:

$$\frac{dy}{y+2} = (x-1) dx$$

Integrate:

$$\begin{aligned}\int \frac{dy}{y+2} &= \int (x-1) dx \\ \ln |y+2| &= \frac{1}{2}x^2 - x + C_1\end{aligned}$$

Isolate y :

$$\begin{aligned}|y+2| &= e^{\frac{1}{2}x^2 - x + C_1} = Ce^{\frac{1}{2}x^2 - x} \\ y(x) &= Ce^{\frac{1}{2}x^2 - x} - 2\end{aligned}$$

- (c) Plug in $x = 0, y = 0$:

$$\begin{aligned}(0) &= Ce^{\frac{1}{2}(0)^2 - (0)} - 2 \\ 0 &= Ce^0 - 2 \\ 0 &= C - 2 \\ C &= 2\end{aligned}$$

So,

$$y(x) = 2e^{\frac{1}{2}x^2 - x} - 2.$$

- (d) The solution is increasing where $\frac{dy}{dx} > 0$.

Using the differential equation:

$$\frac{dy}{dx} = (x-1)(y+2).$$

For the particular solution, $y + 2 = 2e^{\frac{1}{2}x^2 - x}$, which is always positive for all real x .

Therefore, the sign of $\frac{dy}{dx}$ is determined entirely by $x - 1$.

- If $x > 1$, then $x - 1 > 0 \Rightarrow \frac{dy}{dx} > 0$ (increasing).
- If $x < 1$, then $x - 1 < 0 \Rightarrow \frac{dy}{dx} < 0$ (decreasing).

The particular solution is increasing on $(1, \infty)$.